Lösungshinweise zu Fouriertransformation SS 06
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Auf Skizzen wird in den Lösungshinweisen verzichtet.

Aufgabe 1

b) $$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} e^{-\frac{t}{\tau}} e^{-j\omega t}\, dt = \int_{0}^{\infty} e^{-\left(\frac{1}{\tau}+j\omega\right)t}\, dt = \frac{\tau}{1+j\omega\tau}$$

$$\left|F(\omega)\right| = \left|\frac{\tau}{1+j\omega\tau}\right| = \left|\frac{(1-j\omega\tau)\tau}{1+(\omega\tau)^2}\right| = \frac{\tau}{\sqrt{1+(\omega\tau)^2}}$$

Aufgabe 2

b) $$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t}\, dt = \int_{-\infty}^{0} e^{at} e^{-j\omega t}\, dt + \int_{0}^{\infty} e^{-at} e^{-j\omega t}\, dt = \frac{1}{a-j\omega} + \frac{1}{a+j\omega} = \frac{2a}{a^2+\omega^2}$$

Aufgabe 3

a) $$f(t) = \begin{cases} -A & -t_0 < t < 0 \\ A \quad \text{für} & 0 < t < t_0 \\ 0 & |t| > t_0 \end{cases}$$

b) $$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t}\, dt = -\int_{-t_0}^{0} A e^{-j\omega t}\, dt + \int_{0}^{t_0} A e^{-j\omega t}\, dt = \frac{A}{j\omega}(1-e^{j\omega t_0}) - \frac{A}{j\omega}(e^{-j\omega t_0}-1)$$

$$= j\frac{2A}{\omega}(\cos(\omega t_0) - 1)$$

Aufgabe 4

b) $$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t}\, dt = \int_{-T}^{0} \frac{t+T}{T} e^{-j\omega t}\, dt + \int_{0}^{T} \frac{T-t}{T} e^{-j\omega t}\, dt$$

$$\int_{-T}^{0} \frac{t+T}{T} e^{-j\omega t}\, dt = \frac{1}{T}\left[-\frac{1}{j\omega} T e^{j\omega T} - \frac{1}{(j\omega)^2}\left(1-e^{j\omega T}\right)\right] - \frac{1}{j\omega}(1-e^{j\omega T})$$

$$\int_{0}^{T} \frac{T-t}{T} e^{-j\omega t}\, dt = \frac{1}{j\omega} e^{-j\omega T} + \frac{1}{T}\frac{1}{(j\omega)^2}\left(e^{-j\omega T}-1\right) - \frac{1}{j\omega}\left(e^{-j\omega T}-1\right)$$

Damit folgt

$$F(\omega)=\frac{2}{T\omega^2}\left(1-\cos\omega T\right)=\left|F(\omega)\right|$$

c) $$e(\omega)=2\left|F(\omega)\right|^2=\frac{8}{T^2\omega^4}\left(1-\cos\omega T\right)^2$$

Aufgabe 5

b) $$F(\omega)=\int_{-\infty}^{\infty}f(t)e^{-j\omega t}dt=\int_{-\infty}^{\infty}rect(at)e^{-j\omega t}dt=\int_{-\frac{1}{2a}}^{\frac{1}{2a}}e^{-j\omega t}dt=-\frac{1}{j\omega}\left(e^{-\frac{j\omega}{2a}}-e^{\frac{j\omega}{2a}}\right)=\frac{\sin\left(\frac{\omega}{2a}\right)}{\frac{\omega}{2}}$$

Aufgabe 6

a) $$f_1(t)=\begin{cases}0 & \text{für } t<0\\ 1-t & \text{für } 0\le t\le 1\\ 0 & \text{für } t>1\end{cases}$$

$$f_2(t)=\begin{cases}0 & \text{für } t<-1\\ \frac{1}{2}(t+1) & \text{für } -1\le t<0\\ -\frac{1}{2}(t-1) & \text{für } 0\le t<1\\ 0 & \text{für } t\ge 1\end{cases}$$

$$f_3(t)=\begin{cases}0 & \text{für } t<-1\\ -\frac{1}{2}(t+1) & \text{für } -1\le t<0\\ \frac{1}{2}(1-t) & \text{für } 0\le t<1\\ 0 & \text{für } t\ge 1\end{cases}$$

b) Bei $f_2(t)$ handelt es sich um den geraden Anteil von $f_1(t)$. $f_3(t)$ ist der ungerade Anteil von $f_1(t)$.

c) $$F_1(\omega)=\int_{-\infty}^{\infty}f_1(t)e^{-j\omega t}dt=\int_0^1(1-t)e^{-j\omega t}dt=\int_0^1 e^{-j\omega t}dt-\int_0^1 te^{-j\omega t}dt=\frac{1}{j\omega}+\frac{1}{\omega^2}\left(1-e^{-j\omega}\right)$$

$$F_2(\omega)=\int_{-\infty}^{\infty}f_2(t)e^{-j\omega t}dt=\frac{1}{2}\int_{-1}^{0}(t+1)e^{-j\omega t}dt+\frac{1}{2}\int_0^1(-t+1)e^{-j\omega t}dt$$

$$= \frac{1}{\omega^2}(1-\cos\omega)$$

$$F_3(\omega) = \int_{-\infty}^{\infty} f_3(t)e^{-j\omega t}\,dt = -\frac{1}{2}\int_{-1}^{0}(t+1)e^{-j\omega t}\,dt + \frac{1}{2}\int_{0}^{1}(1-t)e^{-j\omega t}\,dt$$

$$= -j\left(\omega - \frac{\sin\omega}{\omega^2}\right)$$

Aufgabe 7

b) $$F_1(\omega) = \int_{-\infty}^{\infty} f_1(t)e^{-j\omega t}\,dt = \int_{-\frac{\pi}{\omega_0}}^{\frac{\pi}{\omega_0}} \cos(\omega_0 t)e^{-j\omega t}\,dt$$

$$= \frac{e^{-j\frac{\omega}{\omega_0}\pi}}{\omega_0{}^2-\omega^2}(\omega_0\sin\pi - j\omega\cos\pi) - \frac{e^{j\frac{\omega}{\omega_0}\pi}}{\omega_0{}^2-\omega^2}(\omega_0\sin(-\pi) - j\omega\cos(-\pi))$$

$$= \frac{2\omega}{\omega_0{}^2-\omega^2}\sin\left(\frac{\omega}{\omega_0}\pi\right) \quad \text{für} \quad \omega \neq \omega_0$$

Für $\omega = \omega_0$ ergibt sich $F_1(\omega) = \frac{\pi}{\omega_0}$.

$$F_2(\omega) = \int_{-\infty}^{\infty} f_2(t)e^{-j\omega t}\,dt = \int_{-\frac{\pi}{\omega_0}}^{\frac{\pi}{\omega_0}} \sin(\omega_0 t)e^{-j\omega t}\,dt$$

$$= \frac{e^{-j\frac{\omega}{\omega_0}\pi}}{\omega_0{}^2-\omega^2}(-j\omega\sin\pi - \omega_0\cos\pi) - \frac{e^{j\frac{\omega}{\omega_0}\pi}}{\omega_0{}^2-\omega^2}(-j\omega\sin(-\pi) - \omega_0\cos(-\pi))$$

$$= -\frac{2j\omega_0}{\omega_0{}^2-\omega^2}\sin\left(\frac{\omega}{\omega_0}\pi\right) \quad \text{für} \quad \omega \neq \omega_0$$

Für $\omega = \omega_0$ ergibt sich $F_2(\omega) = \frac{-j\pi}{\omega_0}$.

Aufgabe 8

a) Für eine N=8 DFT sind die Drehfaktoren

$$W^0 = 1,\; W^1 = e^{-j\frac{\pi}{4}} = \frac{\sqrt{2}}{2} - j\frac{\sqrt{2}}{2},\; W^2 = -j,\; W^3 = e^{-j\frac{3\pi}{4}} = -\frac{\sqrt{2}}{2} - j\frac{\sqrt{2}}{2},$$

$$W^4 = -1,\; W^5 = e^{-j\frac{5\pi}{4}} = -\frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2},\; W^6 = j,\; W^7 = e^{-j\frac{7\pi}{4}} = \frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2}.$$

Für die Matrizen $\underline{\underline{A}}$ und $\underline{\underline{B}}$ findet man

$$\underline{\underline{A}} = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & \frac{\sqrt{2}}{2} - j\frac{\sqrt{2}}{2} & -j & -\frac{\sqrt{2}}{2} - j\frac{\sqrt{2}}{2} & -1 & -\frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2} & j & \frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2} \\
1 & -j & -1 & j & 1 & -j & -1 & j \\
1 & \frac{\sqrt{2}}{2} - j\frac{\sqrt{2}}{2} & j & \frac{\sqrt{2}}{2} - j\frac{\sqrt{2}}{2} & -1 & \frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2} & -j & -\frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2} \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
1 & -\frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2} & -j & \frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2} & -1 & \frac{\sqrt{2}}{2} - j\frac{\sqrt{2}}{2} & j & -\frac{\sqrt{2}}{2} - j\frac{\sqrt{2}}{2} \\
1 & j & -1 & -j & 1 & j & -1 & -j \\
1 & \frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2} & j & -\frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2} & -1 & \frac{\sqrt{2}}{2} - j\frac{\sqrt{2}}{2} & -j & -\frac{\sqrt{2}}{2} - j\frac{\sqrt{2}}{2}
\end{pmatrix}$$

$$\underline{\underline{B}} = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & \frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2} & j & -\frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2} & -1 & -\frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2} & -j & \frac{\sqrt{2}}{2} - j\frac{\sqrt{2}}{2} \\
1 & j & -1 & -j & 1 & j & -1 & -j \\
1 & -\frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2} & -j & \frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2} & -1 & \frac{\sqrt{2}}{2} - j\frac{\sqrt{2}}{2} & j & -\frac{\sqrt{2}}{2} - j\frac{\sqrt{2}}{2} \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
1 & -\frac{\sqrt{2}}{2} - j\frac{\sqrt{2}}{2} & j & \frac{\sqrt{2}}{2} - j\frac{\sqrt{2}}{2} & -1 & \frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2} & -j & -\frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2} \\
1 & -j & -1 & j & 1 & -j & -1 & j \\
1 & \frac{\sqrt{2}}{2} - j\frac{\sqrt{2}}{2} & -j & -\frac{\sqrt{2}}{2} - j\frac{\sqrt{2}}{2} & -1 & -\frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2} & j & \frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2}
\end{pmatrix}$$

b) Die Matrizen $\underline{\underline{A}}$ und $\underline{\underline{B}}$ sind (bis auf einen Faktor) invers zueinander.

c) Der Signalvektor ist $\underline{f}^T = \begin{pmatrix}0 & 1 & 0 & -1 & 0 & 1 & 0 & -1\end{pmatrix}$, das transformierte Signal ist

$$\underline{F_D}^T = \frac{1}{8}\underline{\underline{A}}\,\underline{f} = \begin{pmatrix}0 & 0 & -\frac{j}{2} & 0 & 0 & 0 & \frac{j}{2} & 0\end{pmatrix}^T$$

Aufgabe 9

b)

$$\underline{F_D} = \frac{1}{8}\underline{\underline{A}}\,\underline{f} = \frac{1}{8}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & W & -j & W^3 & -1 & W^5 & j & W^7 \\ 1 & -j & -1 & j & 1 & -j & -1 & j \\ 1 & W^3 & j & W & -1 & W^7 & -j & W^5 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & W^5 & -j & W^7 & -1 & W & j & W^3 \\ 1 & j & -1 & -1 & 1 & j & -1 & -j \\ 1 & W^7 & j & W^5 & -1 & W^3 & -j & W \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 0 \\ -\frac{1}{4}-\frac{j}{4} \\ 0 \\ 0 \\ 0 \\ -\frac{1}{4}+\frac{j}{4} \\ 0 \end{pmatrix}$$

Aufgabe 10

a)

$$F_D(0) = -\frac{1}{8}$$

$$A_1 = 2|F_D(1)| = \frac{1}{4} = A_2 = A_3 \quad A_4 = \frac{7}{4}$$

$$\varphi_1 = -\arg(F_D(1)) = \frac{3\pi}{4}$$

$$\varphi_2 = -\arg(F_D(2)) = \frac{\pi}{2}$$

$$\varphi_3 = -\arg(F_D(3)) = \frac{9\pi}{4}$$

$$\varphi_4 = -\arg(F_D(4)) = 0$$

b) Nein! Aufgrund der Symmetrien sind nur $\frac{N}{2}+1$ Amplituden und Phasen unabhängig.